

Real null coframes in general relativity and GPS type coordinates

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Based on work of Derrick, Coll, and Morales, we define a ‘symmetric’ null coframe with *four real null covectors*. We show that this coframe is closely related to the GPS type coordinates recently introduced by Rovelli [1].

I. INTRODUCTION

In the pseudo-Riemannian spacetime of general relativity theory, we can introduce at each point of spacetime a local vector basis or frame $e_\alpha(x)$, with $\alpha, \beta, \dots = 0, 1, 2, 3$. As already indicated by the notation, the zeroth leg e_0 is conventionally chosen to be timelike whereas e_1, e_2, e_3 are chosen to be spacelike. Specifically, the frame is *pseudo-orthonormal*, i.e., $g(e_\alpha, e_\beta) = \text{diag}(+1, -1, -1, -1) =: o_{\alpha\beta}$. Here $g(u, v) \equiv u \cdot v$ denotes the scalar product of two vectors u and v defined by the Riemannian metric and $o_{\alpha\beta}$ are the components of the Lorentz metric¹.

Three spacelike legs are not particularly practical if one wants to investigate, say, electromagnetic or gravitational wave propagation. Then, on the coordinate level, one introduces null coordinates $\sim t \pm x$ (advanced and retarded ones). Similarly, on the level of the frame, one has two null legs $\sim e_0 \pm e_1$. The two remaining legs are untouched and stay spacelike. Such a *half-null* frame leads immediately to the idea to transform also the space-like legs to null legs. If one wants to uphold orthonormality, this is only possible if one defines the new complex legs $m, \bar{m} \sim e_2 \pm i e_3$, with $i^2 = -1$. This *Newman-Penrose* null frame consists of two real and two complex null vectors [2]. It turned out to be extremely useful for studying the properties of gravitational waves and of exact solutions of Einstein’s field equation.

If one relaxes the constraint of orthonormality, one could also hope that one is able to define a frame consisting of four *real* null vectors. In fact, such a frame was found by Coll & Morales [3], and corresponding coordinates even earlier by Derrick [4], see also [5–8]. Amongst those frames, there are specifically the ‘symmetric null frames’ consisting of four legs none of which is geometrically distinguished. Such a null frame f_α can be constructed from an orthonormal frame by a suitable non-degenerate linear transformation. Since this construction is not widely known, we will present it in Sec.II in a hopefully easily accessible way. We will study some of the properties of that frame and will visualize it by means of a tetrahedron in 3-dimensional space. In Sec.III we will introduce a real null coframe Φ^α which turns out *not* to be dual to the null frame f_α .

Recently, Rovelli [1], in a framework related to the Global Positioning System (GPS) [9], constructed four coordinates s^i in terms of which the components of the metric look null symmetric, see also Coll [10]. In Sec.IV, we follow up these ideas and show that Rovelli’s coframe ds^i is closely related to the real coframe Φ^α .

II. FRAMES CONSISTING OF FOUR REAL NULL VECTORS

At each point of 4-dimensional spacetime with coordinates x^i , here $i, j \dots = 0, 1, 2, 3$, we have the 4-dimensional tangent vector space. Four linearly independent vectors e_α constitute a basis or, alternatively expressed, a *frame*. Dual to this frame is the *coframe* ϑ^β which consists of 4 covectors (one-forms). One can decompose frame and coframe with respect to the local coordinate frame according to

$$e_\alpha = e^i_\alpha \partial_i, \quad \vartheta^\beta = e_j^\beta dx^j, \quad (1)$$

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¹... or, historically correct, the Minkowski metric.

with the duality relations

$$e_i^\alpha e^\beta_\alpha = \delta_i^\beta \quad e_i^\alpha e^\beta_\beta = \delta^\alpha_\beta. \quad (2)$$

The e^i_α are called frame (or tetrad) components.

Let the tangent vector space carry a metric g . Thereby we can define a scalar product $g(u, v) \equiv u \cdot v$, where u and v are two vectors. Accordingly, the components of the metric with respect to the frame e_α are determined by $g_{\alpha\beta} := g(e_\alpha, e_\beta)$. Conversely, the metric can be reconstructed from its components via

$$g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta, \quad (3)$$

see Frankel [11], for instance.

Traditionally, in relativity theory the vectors of an *orthonormal* frame are labeled by $0, 1, 2, 3$, thus underlining the fundamental difference between e_0 , which has positive length $g_{00} = g(e_0, e_0) = 1$, and the e_a , $a = 1, 2, 3$, which have negative lengths $g_{aa} = g(e_a, e_a) = -1$ (no summation). In general, a vector u is called *time-like* if $g(u, u) > 0$, *space-like* if $g(u, u) < 0$, and *null* (or light-like) if $g(u, u) = 0$. The components of the metric tensor with respect to the orthonormal frame e_α are then

$$g_{\alpha\beta} \stackrel{*}{=} o_{\alpha\beta} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = o^{\alpha\beta}. \quad (4)$$

The star equal sign indicates that the corresponding equation is only valid for specific frames, namely for orthonormal ones.

A. Null frames

Starting from an orthonormal frame e_α with respect to which the metric has the standard form (4), we can build a new frame $e_{\alpha'} = (l, n, e_{2'}, e_{3'})$ by the linear transformation

$$l = \frac{1}{\sqrt{2}}(e_0 + e_1), \quad n = \frac{1}{\sqrt{2}}(e_0 - e_1), \quad (5)$$

and $e_{2'} = e_2, e_{3'} = e_3$. In the kernel index method that we are using, see Schouten [12], the frames $e_{\alpha'}$ and e_α are distinguished by the different type of indices. The first two vectors of the new frame are null: $g(l, l) = g(n, n) = 0$. Correspondingly, the metric with respect to this *half-null* frame $e_{\alpha'}$ reads

$$g_{\alpha'\beta'} \stackrel{*}{=} h_{\alpha'\beta'} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = h^{\alpha'\beta'}. \quad (6)$$

Following Newman & Penrose [2], we can further construct two more *null* vectors as the *complex* linear combinations of e_2 and e_3 :

$$m = \frac{1}{\sqrt{2}}(e_2 + i e_3), \quad \bar{m} = \frac{1}{\sqrt{2}}(e_2 - i e_3). \quad (7)$$

Here i is the imaginary unit and overbar means complex conjugation. This transformation leads to the Lorentz metric in a *Newman-Penrose* null frame $e_{\alpha''} = (l, n, m, \bar{m})$:

$$g_{\alpha''\beta''} \stackrel{*}{=} n_{\alpha''\beta''} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = n^{\alpha''\beta''}. \quad (8)$$

Such a frame is convenient for investigating the properties of gravitational and electromagnetic waves.

In the Newman-Penrose frame, we have two real null legs, namely l and n , and two complex ones, m and \bar{m} . It may be surprising to learn that it is also possible to define the *special null* frame which consists of four *real* null vectors. We start from an orthonormal frame e_α , with $g(e_\alpha, e_\beta) = o_{\alpha\beta}$, and define the new frame f_α according to

$$\begin{aligned} f_0 &= (\sqrt{3}e_0 + e_1 + e_2 + e_3)/2, \\ f_1 &= (\sqrt{3}e_0 + e_1 - e_2 - e_3)/2, \\ f_2 &= (\sqrt{3}e_0 - e_1 + e_2 - e_3)/2, \\ f_3 &= (\sqrt{3}e_0 - e_1 - e_2 + e_3)/2. \end{aligned} \tag{9}$$

This can also be written as

$$f_\alpha = e_\beta F_\alpha^\beta, \quad \text{with} \quad F_\alpha^\beta = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 & 1 & 1 \\ \sqrt{3} & 1 & -1 & -1 \\ \sqrt{3} & -1 & 1 & -1 \\ \sqrt{3} & -1 & -1 & 1 \end{pmatrix}. \tag{10}$$

Since $g(f_\alpha, f_\alpha) = 0$ for all α , the null frame consists solely of real non-orthogonal null-vectors. The metric with respect to this frame reads

$$g_{\alpha\beta}^{\sim\sim} = f_{\alpha\beta}^{\sim\sim} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \neq f^{\alpha\beta}. \tag{11}$$

This inequality means that the coframe dual to the frame f_α is *not null*.

B. Properties of real null frames

The metric (11) looks completely symmetric in all its components: Seemingly the time coordinate is not preferred in any sense. Nevertheless, Eq.(11) represents a truly Lorentzian metric. Its determinant is -3 and the eigenvalues are readily computed to be

$$3, \quad -1, \quad -1, \quad -1, \tag{12}$$

which shows that the metric (11) has, indeed, the correct signature. The frame f_α , by the linear transformation $e_\alpha = f_\beta F_\alpha^\beta$, can be transformed back to the orthonormal frame e_α . The matrix F_α^β is inverse to the matrix F_α^β in (10), i.e., $F_\alpha^\beta F_\beta^\gamma = \delta_\alpha^\gamma$.

Provided the original orthonormal frame e_α is a coordinate or *natural* frame, i.e., $e_\alpha = \delta_\alpha^i \partial/\partial x^i$, then, because of (10) and $F_\alpha^\beta = \text{const.}$, the frame f_α is also natural. Accordingly, we can introduce coordinates $\xi^i = (\tau, \xi, \eta, \zeta)$ such that (we drop now the tilde)

$$f_\alpha = \delta_\alpha^i \partial/\partial \xi^i \quad \text{or} \quad f^\alpha = \delta_i^\alpha d\xi^i. \tag{13}$$

Under those circumstances, the metric reads

$$\begin{aligned} g &= f_{\alpha\beta} f^\alpha \otimes f^\beta = f_{ij} d\xi^i d\xi^j \\ &= 2(d\tau d\xi + d\tau d\eta + d\tau d\zeta + d\xi d\eta + d\xi d\zeta + d\eta d\zeta). \end{aligned} \tag{14}$$

There is a beautiful geometrical interpretation of the four null legs of the frame f_α , see Finkelstein [6]. In a *Minkowski spacetime*, let us consider the three-dimensional spacelike hypersurface which is spanned by (e_1, e_2, e_3) . The four points which are defined by the spatial parts of the frame vectors (9), with coordinates $A = \frac{1}{2}(1, 1, 1)$, $B = \frac{1}{2}(1, -1, -1)$, $C = \frac{1}{2}(-1, 1, -1)$, and $D = \frac{1}{2}(-1, -1, 1)$, form a regular tetrahedron in the 3-dimensional subspace, see Fig.1. The vertices A , B , C , and D are equidistant from the origin $O = (0, 0, 0)$; the distance is $\sqrt{3}/2$. Correspondingly, all sides of this tetrahedron have equal length, namely $\sqrt{2}$. If we now send, at the moment $t = 0$, a light pulse from the origin O , it reaches all four vertices of the tetrahedron at $t = \sqrt{3}/2$. Thus four light rays provide the operational definition

for the light-like frame f_α in (9). In a Riemannian spacetime, we can choose Riemannian normal coordinates. Then, provided the tetrahedron is sufficiently small, we will have an analogous interpretation.

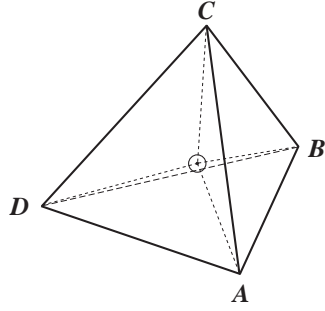


FIG. 1. A tetrahedron which represents the real symmetric null frame in 3-dimensional space. At time $t = 0$, light is emitted in O . It reaches the points A, B, C , and D at $t = \sqrt{3}/2$. The event $(0, O)$, together with the events $(\sqrt{3}/2, A)$, $(\sqrt{3}/2, B)$, etc., determine the four null vectors.

In the *special* real null frame (9), all legs are equivalent. However, it is possible to apply a linear transformation that keeps the zeros in the diagonal of (11) but changes the off diagonal matrix elements such that the matrix remains symmetric and non-degenerate. In this way, we find a *general* real null frame. Under such circumstances, the four real null vectors f_α are no longer ‘indistinguishable’.

III. COFRAMES CONSISTING OF FOUR REAL NULL COVECTORS

The inverse of the matrix (11) is *not* the same matrix, in contrast to the analogous cases (4), (6), and (8). We rather find

$$f^{\alpha\beta} = \frac{1}{3} \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}. \quad (15)$$

In other words, $f_{\alpha\alpha} = 0$, but $f^{\alpha\alpha} \neq 0$ (no summation over α). However, we are able to find a new real null *coframe* by defining, in analogy to (9),

$$\begin{aligned} \Phi^{0'} &= (\sqrt{3}\vartheta^0 + \vartheta^1 + \vartheta^2 + \vartheta^3)/2, \\ \Phi^{1'} &= (\sqrt{3}\vartheta^0 + \vartheta^1 - \vartheta^2 - \vartheta^3)/2, \\ \Phi^{2'} &= (\sqrt{3}\vartheta^0 - \vartheta^1 + \vartheta^2 - \vartheta^3)/2, \\ \Phi^{3'} &= (\sqrt{3}\vartheta^0 - \vartheta^1 - \vartheta^2 + \vartheta^3)/2. \end{aligned} \quad (16)$$

Accordingly, we have

$$\Phi^{\alpha'} = B_{\beta}^{\alpha'} \vartheta^{\beta}, \quad \text{with} \quad B_{\beta}^{\alpha'} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad (17)$$

where B is the transpose of F in (10): $B = F^T$. Then, we obtain

$$g^{\alpha'\beta'} = B_{\mu}^{\alpha'} B_{\nu}^{\beta'} o^{\mu\nu} = (B^T o B)^{\alpha'\beta'} = \Phi^{\alpha'\beta'} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \quad (18)$$

Thus, $\Phi^{\alpha'}$ is a *special* null coframe with the contravariant metric components as given by (18).

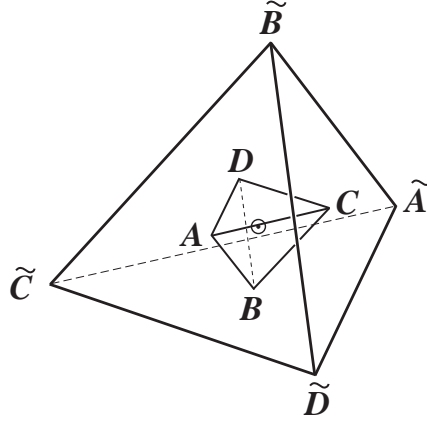


FIG. 2. The ‘outer’ tetrahedron represents the real null coframe of (16) with the contravariant metric components (18). The ‘inner’ tetrahedron is that of Fig.1. Note that the 3-vectors OA, OB , etc. are perpendicular to the planes $\tilde{B}\tilde{C}\tilde{D}$, $\tilde{C}\tilde{D}\tilde{A}$, etc.

The visualization of the new null coframe leads also to a tetrahedron in 3-space which is “dual” to the one of Fig.1. We depicted that in Fig.2. The vertices of the coframe tetrahedron are $\tilde{A} = -\frac{3}{2}(1, 1, 1)$, $\tilde{B} = -\frac{3}{2}(1, -1, -1)$, $\tilde{C} = -\frac{3}{2}(-1, 1, -1)$, and $\tilde{D} = -\frac{3}{2}(-1, -1, 1)$. Its sides have a length of $3\sqrt{2}$. The barycenters of the four triangles of the tetrahedron are the vertices A, B, C, D of the tetrahedron of Fig.1.

According to the construction, the 3-vectors OA, OB, OC , and OD , representing light rays, are perpendicular to the triangles $\tilde{B}\tilde{C}\tilde{D}$, $\tilde{C}\tilde{D}\tilde{A}$, $\tilde{D}\tilde{A}\tilde{B}$, and $\tilde{A}\tilde{B}\tilde{C}$, respectively. Therefore, at the points A etc., the planes $\tilde{B}\tilde{C}\tilde{D}$ etc. represent the wave fronts of the light. Together with the parallel planes through $O = (0, 0, 0)$, the planes $\tilde{B}\tilde{C}\tilde{D}$ etc. symbolize covectors that are algebraically dual to the vectors OA etc. The interior product of the vectors with the corresponding covectors is always 1. In this sense, the outer tetrahedron is dual to the inner one.

The most general null coframe has the form

$$\Phi^{\bar{\alpha}} = B_{\beta}^{\bar{\alpha}} \vartheta^{\beta}, \quad \text{with} \quad B_{\beta}^{\bar{\alpha}} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \end{pmatrix}, \quad (19)$$

where $\alpha^2 := \alpha_{\mu}\alpha_{\nu} o^{\mu\nu} = \beta^2 = \gamma^2 = \delta^2 = 0$ and $\det B \neq 0$. The corresponding contravariant components of the metric read

$$g^{\bar{\alpha}\bar{\beta}} = \begin{pmatrix} 0 & \alpha \cdot \beta & \alpha \cdot \gamma & \alpha \cdot \delta \\ \beta \cdot \alpha & 0 & \beta \cdot \gamma & \beta \cdot \delta \\ \gamma \cdot \alpha & \gamma \cdot \beta & 0 & \gamma \cdot \delta \\ \delta \cdot \alpha & \delta \cdot \beta & \delta \cdot \gamma & 0 \end{pmatrix}. \quad (20)$$

For $B_{\beta}^{\bar{\alpha}} = B_{\beta}^{\alpha'}$, the general expression for $g^{\bar{\alpha}\bar{\beta}}$ reduces to the special form (18).

IV. ROVELLI’S CONSTRUCTION AND REAL NULL COFRAMES

Let us eventually turn to Rovelli’s paper [1] which prompted our remarks in the first place. The 4-velocity of a massive particle moving in a (flat) Minkowski space M_4 , expressed in the standard *inertial* coordinates x^i , has the form

$$W^i := \frac{dx^i}{ds} = \frac{1}{\sqrt{1-v^2}} (1, v^a), \quad (21)$$

where $v^a := dx^a/dt = v n^a$ represents the 3-velocity and n^a a unit 3-vector. We assume v to be constant. The 4-vector W^i is normalized, i.e., $W^2 := W \cdot W = 1$. For constant W^i , the particle’s trajectory is of the simple form

$$x^i(s) = s W^i. \quad (22)$$

Suppose there is an observer at some point P with coordinates $x^i = X^i$ who is able to detect light signals emitted from the moving particle. The observer's light cone is described by the equation

$$(x - X)^2 = 0. \quad (23)$$

The condition that the particle's trajectory intersects this light cone is obtained by solving the system of equations (22) and (23). Substituting (22) into (23), we obtain a quadratic equation for s , namely $s^2 - 2s W \cdot X + X^2 = 0$, which yields

$$s = W \cdot X - \sqrt{(W \cdot X)^2 - X^2}. \quad (24)$$

The minus sign given in front of the square root refers to the past light cone, whereas the plus sign is suppressed since it refers to the future light cone. We should note that the result in (24) is correct for *any* choice of the unit vector n in W .

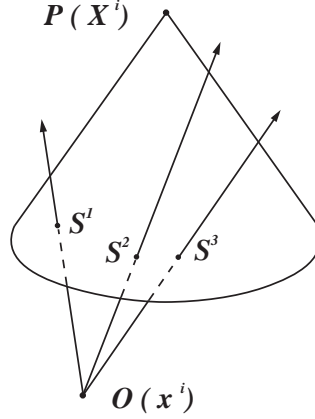


FIG. 3. This is a 1 + 2-dimensional spacetime diagram, i.e., one spatial coordinate is suppressed. The diagram represents Rovelli's satellites which supply the 'distance' coordinates s^1, s^2, s^3 for the event $P(X^i)$. In 1+3 dimensions, we have, of course, *four* observers.

If we consider a set of four particles, see Fig.3, in which one space dimension is suppressed and thus only three particles are visible, then Eqs.(21) and (24) are replaced by equations of the same type,

$$W_k^{i'} = \frac{1}{\sqrt{1 - (v^{i'})^2}} \left(1, v^{i'} n_a^{i'} \right), \quad s^{i'} = W^{i'} \cdot X - \sqrt{(W^{i'} \cdot X)^2 - X^2}. \quad (25)$$

Here $W^{i'} \cdot X := W_k^{i'} X^k$ and $i' = 0, 1, 2, 3$ is a label for different particles. Note that the four 3-velocities $v^{i'}$ are in general different from each other, in contrast to the original Rovelli construction where they are all equal. This generalization is physically quite natural, and it will be important in our considerations. In the next step, following Rovelli, we introduce $s^{i'}$ as new coordinates on the M_4 . The Jacobian matrix of the transformation $x^k \rightarrow s^{i'}$ at the observer's point $P(X)$ is given by

$$E_k^{i'} = \frac{\partial s^{i'}}{\partial x^k}(x = X) = \frac{\partial s^{i'}}{\partial X^k} = W_k^{i'} - \frac{W_k^{i'}(W^{i'} \cdot X) - X_k}{\sqrt{(W^{i'} \cdot X)^2 - X^2}}. \quad (26)$$

The contravariant components of the metric in the new coordinates are

$$\begin{aligned} g^{i'j'} &= E_k^{i'} E_l^{j'} o^{kl} \\ &= W^{i'} \cdot W^{j'} - \left[\frac{(W^{i'} \cdot W^{j'})(W^{j'} \cdot X) - (W^{i'} \cdot X)(W^{j'} \cdot X)}{\sqrt{(W^{j'} \cdot X)^2 - X^2}} + (i' \leftrightarrow j') \right] \\ &\quad + \frac{(W^{i'} \cdot W^{j'})(W^{i'} \cdot X)(W^{j'} \cdot X) - (W^{i'} \cdot X)^2 - (W^{j'} \cdot X)^2 + X^2}{\sqrt{(W^{i'} \cdot X)^2 - X^2} \sqrt{(W^{j'} \cdot X)^2 - X^2}}. \end{aligned} \quad (27)$$

A direct calculation for $j' = i'$, by using $(W^{i'})^2 = 1$, leads to

$$g^{i'j'} = \frac{\partial s^{i'}}{\partial x^k} \frac{\partial s^{j'}}{\partial x^l} o^{kl} = 0 \quad (i', j' = 0, 1, 2, 3). \quad (28)$$

Note that this result is valid for *any* choice of the unit vectors $n^{i'}$ in $W^{i'}$. The components $g^{i'j'}$, for $i' \neq j'$, depend only on the scalar products $W^{i'} \cdot W^{j'}$ and $W^{i'} \cdot X$. The surfaces $s^{i'} = \text{const}$ are null hypersurfaces. We now drop the primes for simplicity.

In *frame* (9), we have $g_{\alpha\alpha} = 0$, but $g^{\alpha\alpha} \neq 0$, in contrast to equation (28). However, for the general real null coframe of (19), we have the desired $g^{\alpha\alpha} = 0$. Therefore it is evident that the Rovelli coordinates are closely related to this general real null coframe.

Rovelli's coframe $E^\alpha = E_k^\alpha \vartheta^k$ belongs to the class of general null coframes (19), hence it can be reduced to our special null coframe (17) by demanding

$$E_k^\alpha = B_k^\alpha \quad (29)$$

or, more explicitly,

$$\begin{aligned} W_k^0 - \frac{W_k^0(W^0 \cdot X) - X_k}{\sqrt{(W^0 \cdot X)^2 - X^2}} &= (\sqrt{3}, 1, 1, 1)/2, \\ W_k^1 - \frac{W_k^1(W^1 \cdot X) - X_k}{\sqrt{(W^1 \cdot X)^2 - X^2}} &= (\sqrt{3}, 1, -1, -1)/2, \\ W_k^2 - \frac{W_k^2(W^2 \cdot X) - X_k}{\sqrt{(W^2 \cdot X)^2 - X^2}} &= (\sqrt{3}, -1, 1, -1)/2, \\ W_k^3 - \frac{W_k^3(W^3 \cdot X) - X_k}{\sqrt{(W^3 \cdot X)^2 - X^2}} &= (\sqrt{3}, -1, -1, 1)/2. \end{aligned} \quad (30)$$

We shall now find a particular solution of these equations for W_k^α , for a given position X of the observer. Without loss of generality, we can choose the inertial coordinates x^i in such a way that

$$X^i = (\mathcal{T}, \mathcal{X}, 0, 0), \quad \mathcal{T} > 0, \mathcal{X} \neq 0. \quad (31)$$

In that case, by multiplying Eqs.(30) with X^k , we find

$$\begin{aligned} (W^0 \cdot X) - \sqrt{(W^0 \cdot X)^2 - X^2} &= (\sqrt{3}\mathcal{T} + \mathcal{X})/2 =: b^0, \\ (W^1 \cdot X) - \sqrt{(W^1 \cdot X)^2 - X^2} &= (\sqrt{3}\mathcal{T} + \mathcal{X})/2 =: b^1, \\ (W^2 \cdot X) - \sqrt{(W^2 \cdot X)^2 - X^2} &= (\sqrt{3}\mathcal{T} - \mathcal{X})/2 =: b^2, \\ (W^3 \cdot X) - \sqrt{(W^3 \cdot X)^2 - X^2} &= (\sqrt{3}\mathcal{T} - \mathcal{X})/2 =: b^3. \end{aligned} \quad (32)$$

Since $b^0 = b^1$ and $b^2 = b^3$, it follows that

$$W^0 \cdot X = W^1 \cdot X, \quad W^2 \cdot X = W^3 \cdot X. \quad (33)$$

This result combined with Eqs.(30) leads to

$$\begin{aligned} W_0^0 &= W_0^1, & W_1^0 &= W_1^1, & W_2^0 &= -W_2^1, & W_3^0 &= -W_3^1, \\ W_0^2 &= W_0^3, & W_1^2 &= W_1^3, & W_2^2 &= -W_2^3, & W_3^2 &= -W_3^3. \end{aligned} \quad (34)$$

A simple form of W_k^α , consistent with these conditions, is given by

$$\begin{aligned} W_k^0 &= \frac{1}{\sqrt{1-v^2}} (1, -v n^{0a}), \\ W_k^1 &= \frac{1}{\sqrt{1-v^2}} (1, -v n^{1a}), \\ W_k^2 &= \frac{1}{\sqrt{1-u^2}} (1, -u n^{2a}), \\ W_k^3 &= \frac{1}{\sqrt{1-u^2}} (1, -u n^{3a}), \end{aligned} \quad (35)$$

where v and u are undetermined velocities and the n^α are the 3-dimensional unit vectors of (9):

$$\begin{aligned} n^{0a} &= (1, 1, 1)/\sqrt{3}, & n^{1a} &= (1, -1, -1)/\sqrt{3}, \\ n^{2a} &= (-1, 1, -1)/\sqrt{3}, & n^{3a} &= (-1, -1, 1)/\sqrt{3}. \end{aligned} \quad (36)$$

We now use Eqs.(32) in order to express $W^\alpha \cdot X$ in terms of X^i :

$$W^\alpha \cdot X = \frac{(b^\alpha)^2 + X^2}{2b^\alpha} =: h^\alpha. \quad (37)$$

Then we substitute W^α from Eqs.(35) into (37), and obtain (only) two independent equations:

$$\begin{aligned} \frac{1}{\sqrt{1-v^2}} \left(\mathcal{T} - \frac{1}{\sqrt{3}} v \mathcal{X} \right) &= h^0, \\ \frac{1}{\sqrt{1-u^2}} \left(\mathcal{T} + \frac{1}{\sqrt{3}} u \mathcal{X} \right) &= h^2. \end{aligned} \quad (38)$$

Taking into account the explicit expressions for h^0 and h^2 , these equations can be written in the simple form

$$\begin{aligned} a(v) \mathcal{T}^2 + b(v) \mathcal{T} \mathcal{X} + c(v) \mathcal{X}^2 &= 0, \\ a(u) \mathcal{T}^2 - b(u) \mathcal{T} \mathcal{X} + c(u) \mathcal{X}^2 &= 0, \end{aligned} \quad (39)$$

with

$$\begin{aligned} a(w) &:= 4\sqrt{3} - 7\sqrt{1-w^2}, \\ b(w) &:= 4(1-w) - 2\sqrt{3}\sqrt{1-w^2}, \\ c(w) &:= 3\sqrt{1-w^2} - 4w/\sqrt{3}, \end{aligned} \quad (40)$$

for $w = v$ or u . We demand that Eqs.(39) have real solutions for \mathcal{T} and \mathcal{X} . It implies the following conditions on the velocities:

$$\Delta(w) := b^2(w) - 4a(w)c(w) \geq 0. \quad (41)$$

The calculations presented in Fig.4 show the allowed values for the velocities w .

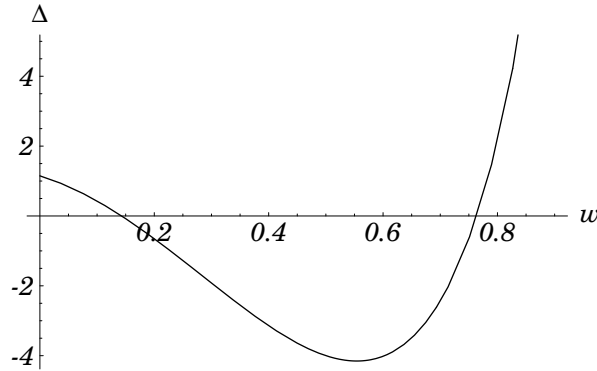


FIG. 4. The allowed values of w are determined by $\Delta(w) \geq 0$.

A particular example of an acceptable set of parameters is given by $\{\mathcal{T} = 1/\sqrt{3}, \mathcal{X} = 4, v = 0.77, u = 0.83\}$. In general, for all sets of parameters $\{\mathcal{T}, \mathcal{X}, v, u\}$ which satisfy the requirements above, Rovelli's null coframe reduces to our special null coframe (17).

In conclusion, in this paper we introduced and discussed the special real null coframe (17) whereupon the meaning of Rovelli's coordinates in Minkowski spacetime becomes clearer. In the Riemannian space of general relativity, Rovelli's coordinates are related to the class of general null coframes (19).

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